BLM REALIZATION FOR FROBENIUS–LUSZTIG KERNELS OF TYPE A

QIANG FU

ABSTRACT. The infinitesimal quantum $\mathfrak{gl}_n$ was realized in [1, §6]. We will realize Frobenius–Lusztig Kernels of type A in this paper.

1. INTRODUCTION

In 1990, Ringel discovered the Hall algebra realization [19] of the positive part of the quantum enveloping algebras of finite type. Almost at the same time, the entire quantum $\mathfrak{gl}_n$ was realized by A. A. Beilinson, G. Lusztig and R. MacPherson in [1]. They first used q-Schur algebras to construct a $\mathbb{Q}(v)$-algebra $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$, and then proved that the quantum enveloping algebra of $\mathfrak{gl}_n$ over $\mathbb{Q}(v)$ can be realized as a subalgebra of $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$.

Let $U_\Bbbk(n)$ be the the quantum enveloping algebra of $\mathfrak{gl}_n$ over $\Bbbk$ with standard generators $E_i^{(m)}$, $F_i^{(m)}$, $K_i^{\pm 1}$ and $\left[{K_i;0 \atop t} \right]$, where $\Bbbk$ is a commutative ring containing a primitive l'th root ε of 1. Let $p = \mathrm{char}\,\Bbbk$. For $h \geqslant 1$, let $\widetilde{\mathfrak{u}}_\Bbbk(n)_h$ be the $\Bbbk$-subalgebra of $U_\Bbbk(n)$ generated by $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm 1}$, $\left[{K_j;0 \atop t} \right]$ for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$ and $0 \leqslant m, t < lp^{h-1}$, where $l = l'$ if l' is odd, and $l = l'/2$ otherwise. Then we have $\widetilde{\mathfrak{u}}_\Bbbk(n)_1 \subseteq \widetilde{\mathfrak{u}}_\Bbbk(n)_2 \subseteq \cdots \subseteq U_\Bbbk(n)$. In the case where l' is an odd number, let $\mathfrak{u}_\Bbbk(n)_h = \widetilde{\mathfrak{u}}_\Bbbk(n)_h / \langle K_1^l - 1, \cdots, K_n^l - 1 \rangle$. The algebra $\mathfrak{u}_\Bbbk(n)_1$ is called the infinitesimal quantum $\mathfrak{gl}_n$ and the algebra $\mathfrak{u}_\Bbbk(n)_h$ is called Frobenius–Lusztig Kernels of $U_\Bbbk(n)$ (cf. [7]). The algebra $\mathfrak{u}_\Bbbk(n)_1$ was realized in [1, §6]. In this paper, we will realize the algebra $\mathfrak{u}_\Bbbk(n)_h$ for all $h \geqslant 1$. More precisely, we will first construct the $\Bbbk$-algebra $\mathscr{K}'(n)_h$ in §4. Then we will prove in 5.5 that $\mathfrak{u}_\Bbbk(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is odd, and that $\widetilde{\mathfrak{u}}_\Bbbk(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is even and $\Bbbk$ is a field.

Let $\mathcal{S}_\Bbbk(n,r)$ be the q-Schur algebra over $\Bbbk$. Certain subalgebra, denoted by $\widetilde{\mathfrak{u}}_\Bbbk(n,r)_h$, of $\mathcal{S}_\Bbbk(n,r)$ was constructed in [12, §4]. It is proved in [13] that $\widetilde{\mathfrak{u}}_\Bbbk(n,r)_1$ is isomorphic to the little q-Schur algebra introduced in [11, 14]. We will prove in 6.1 that the algebra $\widetilde{\mathfrak{u}}_\Bbbk(n,r)_h$ is a homomorphic image of $\widetilde{\mathfrak{u}}_\Bbbk(n)_h$.

Infinitesimal q-Schur algebras are certain important subalgebras of q-Schur algebras (cf. [6, 2, 3]). For $h \geqslant 1$ let $\mathbf{s}_\Bbbk(n)_h$ be the $\Bbbk$-subalgebra of $U_\Bbbk(n)$ generated by the algebra $\widetilde{\mathfrak{u}}_\Bbbk(n)_h$ and $\left[{K_j;0 \atop t} \right]$ $(1 \leqslant j \leqslant n,\, t \in \mathbb{N})$. We will prove in 6.4 that the infinitesimal q-Schur algebra $\mathbf{s}_\Bbbk(n,r)_h$ is a homomorphic image of $\mathbf{s}_\Bbbk(n)_h$.

Throughout this paper, let $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$ where v is an indeterminate and let $\mathcal{Q} = \mathbb{Q}(v)$ be the fraction field of $\mathcal{Z}$. For $i \in \mathbb{Z}$ let $[i] = \frac{v^i - v^{-i}}{v - v^{-1}}$. For integers N, t with $t \geqslant 0$, let

$$\begin{bmatrix} N \\ t \end{bmatrix} = \frac{[N][N-1] \cdots [N-t+1]}{[t]!} \in \mathcal{Z}$$

where $[t]! = [1][2] \cdots [t]$. For $\mu \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ let $\begin{bmatrix} \mu \\ \lambda \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \lambda_1 \end{bmatrix} \cdots \begin{bmatrix} \mu_n \\ \lambda_n \end{bmatrix}$.

Let $\Bbbk$ be a commutative ring containing a primitive l'th root ε of 1 with $l' \geqslant 1$. Let $l \geqslant 1$ be defined by

$$l = \begin{cases} l' & \text{if } l' \text{ is odd,} \\ l'/2 & \text{if } l' \text{ is even.} \end{cases}$$

Let p be the characteristic of $\Bbbk$. We will regard $\Bbbk$ as a $\mathcal{Z}$-module by specializing v to ε. When v is specialized to ε, $\begin{bmatrix} c \\ t \end{bmatrix}$ specialize to the element $\begin{bmatrix} c \\ t \end{bmatrix}_\varepsilon$ in $\Bbbk$.

2. THE BLM CONSTRUCTION OF QUANTUM $\mathfrak{gl}_n$

Following [16] we define the quantum enveloping algebra $U_{\mathcal{Q}}(n)$ of $\mathfrak{gl}_n$ to be the $\mathbb{Q}(v)$-algebra with generators

$$E_i, \ F_i \quad (1 \leqslant i \leqslant n-1), \ K_j, \ K_j^{-1} \quad (1 \leqslant j \leqslant n)$$

and relations

(a) $K_i K_j = K_j K_i, \ K_i K_i^{-1} = 1$;
(b) $K_i E_j = v^{\delta_{i,j} - \delta_{i,j+1}} E_j K_i$;
(c) $K_i F_j = v^{\delta_{i,j+1} - \delta_{i,j}} F_j K_i$;
(d) $E_i E_j = E_j E_i, \ F_i F_j = F_j F_i$ when $|i - j| > 1$;
(e) $E_i F_j - F_j E_i = \delta_{i,j} \frac{\widetilde{K}_i - \widetilde{K}_i^{-1}}{v - v^{-1}}$, where $\widetilde{K}_i = K_i K_{i+1}^{-1}$;
(f) $E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0$ when $|i - j| = 1$;
(g) $F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0$ when $|i - j| = 1$.

Following [17], let $U_{\mathcal{Z}}(n)$ be the $\mathcal{Z}$-subalgebra of $U_{\mathcal{Q}}(n)$ generated by all $E_i^{(m)}$, $F_i^{(m)}$, $K_i^{\pm 1}$ and $\begin{bmatrix} K_i; 0 \\ t \end{bmatrix}$, where for $m, t \in \mathbb{N}$,

$$E_i^{(m)} = \frac{E_i^m}{[m]!}, \ F_i^{(m)} = \frac{F_i^m}{[m]!}, \ \text{and} \ \begin{bmatrix} K_i; 0 \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{-s+1} - K_i^{-1} v^{s-1}}{v^s - v^{-s}}.$$

Let $\Theta(n)$ be the set of all $n \times n$ matrices over $\mathbb{N}$. Let $\Theta^{\pm}(n)$ be the set of all $A \in \Theta(n)$ whose diagonal entries are zero. Let $\Theta^+(n)$ (resp. $\Theta^-(n)$) be the subset of $\Theta(n)$ consisting of those matrices $(a_{i,j})$ with $a_{i,j} = 0$ for all $i \geqslant j$ (resp. $i \leqslant j$). For $A \in \Theta^{\pm}(n)$, write $A = A^+ + A^-$ with $A^+ \in \Theta^+(n)$ and $A^- \in \Theta^-(n)$. For $A \in \Theta^{\pm}(n)$ let

$$E^{(A^+)} = \prod_{\substack{i \leqslant s < j \\ 1 \leqslant i, j \leqslant n}} E_s^{(a_{ij})}, \quad F^{(A^-)} = \prod_{\substack{j \leqslant s < i \\ 1 \leqslant i, j \leqslant n}} F_s^{(a_{i,j})}$$

where the ordering of the products is the same as in [1, 3.9]. According to [17, 4.5] and [18, 7.8] we have the following result.

Proposition 2.1. *The set*

$$\{E^{(A^+)} \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix} F^{(A^-)} \mid A \in \Theta^{\pm}(n),\, \delta, \lambda \in \mathbb{N}^n,\, \delta_i \in \{0, 1\},\, \forall i\}$$

forms a $\mathcal{Z}$-basis of $U_{\mathcal{Z}}(n)$.

Using the stabilization property of the multiplication of q-Schur algebras, an important algebra $\mathcal{K}_{\mathcal{Z}}(n)$ over $\mathcal{Z}$ (without 1), with basis $\{[A] \mid A \in \widetilde{\Theta}(n)\}$ was constructed in [1, 4.5], where $\widetilde{\Theta}(n) = \{(a_{ij}) \in M_n(\mathbb{Z}) \mid a_{ij} \geqslant 0\ \forall 1 \leqslant i \neq j \leqslant n\}$.

Following [1, 5.1], let $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ be the vector space of all formal $\mathbb{Q}(v)$-linear combinations $\sum_{A \in \widetilde{\Theta}(n)} \beta_A[A]$ satisfying the following property: for any $\mathbf{x} \in \mathbb{Z}^n$,

(2.1.1)   the sets $\begin{array}{l} \{A \in \widetilde{\Theta}(n) \mid \beta_A \neq 0,\ \mathrm{ro}(A) = \mathbf{x}\} \\ \{A \in \widetilde{\Theta}(n) \mid \beta_A \neq 0,\ \mathrm{co}(A) = \mathbf{x}\} \end{array}$ are finite,

where $\mathrm{ro}(A) = (\sum_j a_{1,j}, \cdots, \sum_j a_{n,j})$ and $\mathrm{co}(A) = (\sum_i a_{i,1}, \cdots, \sum_i a_{i,n})$ are the sequences of row and column sums of A. The product of two elements $\sum_{A \in \widetilde{\Theta}(n)} \beta_A[A]$, $\sum_{B \in \widetilde{\Theta}(n)} \gamma_B[B]$ in $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ is defined to be $\sum_{A,B} \beta_A \gamma_B[A] \cdot [B]$ where $[A] \cdot [B]$ is the product in $\mathcal{K}_{\mathcal{Z}}(n)$. Then $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ becomes an associative algebra over $\mathbb{Q}(v)$.

For $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ let

$$A(\delta, \lambda) = \sum_{\mu \in \mathbb{Z}^n} v^{\mu \cdot \delta} \begin{bmatrix} \mu \\ \lambda \end{bmatrix} [A + \mathrm{diag}(\mu)] \in \widehat{\mathcal{K}}_{\mathcal{Q}}(n);$$

$$A(\delta) = \sum_{\mu \in \mathbb{Z}^n} v^{\mu \cdot \delta} [A + \mathrm{diag}(\mu)] \in \widehat{\mathcal{K}}_{\mathcal{Q}}(n),$$

where $\mu \cdot \delta = \sum_{1 \leqslant i \leqslant n} \mu_i \delta_i$.

The next result is proved in [1, 5.5, 5.7].

Theorem 2.2. *There is an injective algebra homomorphism $\varphi : U_{\mathcal{Q}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ satisfying*

$$E_i \mapsto E_{i,i+1}(\mathbf{0}),\ \ K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n} \mapsto 0(\mathbf{j}),\ \ F_i \mapsto E_{i+1,i}(\mathbf{0}).$$

Furthermore the set $\{A(\mathbf{j}) \mid A \in \Theta^{\pm}(n),\ \mathbf{j} \in \mathbb{Z}^n\}$ forms a $\mathbb{Q}(v)$-basis for $\varphi(U_{\mathcal{Q}}(n))$.

We shall identify $U_{\mathcal{Q}}(n)$ with $\varphi(U_{\mathcal{Q}}(n))$. According to [15, 4.2,4.3,4.4], we have the following result.

Proposition 2.3. *The algebra $U_{\mathcal{Z}}(n)$ is generated as a $\mathcal{Z}$-module by the elements $A(\delta, \lambda)$ for $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$. Furthermore, each of the following set forms a $\mathcal{Z}$-basis for $U_{\mathcal{Z}}(n)$:*

(1) $\{A(\mathbf{0})0(\delta,\lambda) \mid A \in \Theta^{\pm}(n), \ \delta,\lambda \in \mathbb{N}^n, \ \delta_i \in \{0,1\}, \forall i\}$;

(2) $\{A(\delta,\lambda) \mid A \in \Theta^{\pm}(n), \ \delta,\lambda \in \mathbb{N}^n, \ \delta_i \in \{0,1\}, \forall i\}$.

We end this section by recalling an important triangular relation in $\mathcal{K}_{\mathcal{Z}}(n)$. For $A = (a_{s,t}) \in \widetilde{\Theta}(n)$ let

$$
\sigma_{i,j}(A) = \begin{cases} \sum_{s \leqslant i; t \geqslant j} a_{s,t} & \text{if } i < j \\ \sum_{s \geqslant i; t \leqslant j} a_{s,t} & \text{if } i > j. \end{cases}
$$

Following [1], for $A, B \in \widetilde{\Theta}(n)$, define $B \preccurlyeq A$ if and only if $\sigma_{i,j}(B) \leqslant \sigma_{i,j}(A)$ for all $i \neq j$. Put $B \prec A$ if $B \preccurlyeq A$ and $\sigma_{i,j}(B) < \sigma_{i,j}(A)$ for some $i \neq j$.

According to [1, 5.5(c)], for $A \in \Theta^{\pm}(n)$ and $\lambda \in \mathbb{Z}^n$ the following triangular relation holds in $\mathcal{K}_{\mathcal{Z}}(n)$:

(2.3.1)
$$
E^{(A^+)}[\text{diag}(\lambda)]F^{(A^-)} = [A + \text{diag}(\lambda - \boldsymbol{\sigma}(A))] + f
$$

where $\boldsymbol{\sigma}(A) = (\sigma_1(A), \cdots, \sigma_n(A))$ with $\sigma_i(A) = \sum_{j<i}(a_{i,j} + a_{j,i})$ and f is a finite $\mathcal{Z}$-linear combination of $[B]$ with $B \in \widetilde{\Theta}(n)$ such that $B \prec A$.

3. THE ALGEBRA $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h$

Let $U_{\Bbbk}(n) = U_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} \Bbbk$. We shall denote the images of $E_i^{(m)}$, $F_i^{(m)}$, $A(\delta,\lambda)$, etc. in $U_{\Bbbk}(n)$ by the same letters. For $h \geqslant 1$ let $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h$ be the $\Bbbk$-subalgebra of $U_{\Bbbk}(n)$ generated by the elements $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm 1}$, $\begin{bmatrix} K_j;0 \\ t \end{bmatrix}$ for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$ and $0 \leqslant m,t < lp^{h-1}$. If l' is an odd number, we let

(3.0.2)
$$
\mathfrak{u}_{\Bbbk}(n)_h = \widetilde{\mathfrak{u}}_{\Bbbk}(n)_h / \langle K_1^l - 1, \cdots, K_n^l - 1 \rangle.
$$

The algebra $\mathfrak{u}_{\Bbbk}(n)_h$ is called Frobenius–Lusztig Kernels of $U_{\Bbbk}(n)$. We will construct several $\Bbbk$-bases for $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h$ in 3.7.

We need some preparation before proving 3.7.

Lemma 3.1. *Let $m = m_0 + lm_1$, $0 \leqslant m_0 \leqslant l-1$, $m_1 \in \mathbb{N}$. Then*

$$
\begin{bmatrix} m \\ t \end{bmatrix}_{\varepsilon} = \varepsilon^{l(t_1 l - t_1 m_0 - t m_1)} \begin{bmatrix} m_0 \\ t_0 \end{bmatrix}_{\varepsilon} \begin{pmatrix} m_1 \\ t_1 \end{pmatrix}
$$

for $0 \leqslant t \leqslant m$, where $t = t_0 + lt_1$ with $0 \leqslant t_0 \leqslant l-1$ and $t_1 \in \mathbb{N}$.

Lemma 3.2. *The following identity hold in the field $\Bbbk$: $\binom{m+p^{h-1}}{s} = \binom{m}{s}$ for $m \in \mathbb{Z}$ and $0 \leqslant s < p^{h-1}$.*

Proof. We consider the polynomial ring $\Bbbk[x,y]$. Since the characteristic of $\Bbbk$ is p we see that

$$
\sum_{0 \leqslant j \leqslant p^{h-1}} \binom{p^{h-1}}{j} x^j y^{p^{h-1}-j} = (x+y)^{p^{h-1}} = x^{p^{h-1}} + y^{p^{h-1}}.
$$

It follows that $\binom{p^{h-1}}{j} = 0$ for $0 < j < p^{h-1}$. This implies that

$$\binom{m + p^{h-1}}{s} = \sum_{0 \leqslant j \leqslant s} \binom{p^{h-1}}{j}\binom{m}{s-j} = \binom{m}{s}$$

for $m \in \mathbb{Z}$ and $0 \leqslant s < p^{h-1}$. $\qquad\square$

We now generalize 3.2 to the quantum case.

Lemma 3.3. *Assume $0 \leqslant a < lp^{h-1}$ and $b \in \mathbb{Z}$. Then we have $\begin{bmatrix} b+lp^{h-1} \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon$. In particular, we have $\begin{bmatrix} b+l'p^{h-1} \\ a \end{bmatrix}_\varepsilon = \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon$*

Proof. We write $a = a_0 + a_1 l$ and $b = b_0 + b_1 l$ with $0 \leqslant a_0, b_0 < l$, $a_1 \in \mathbb{N}$ and $b_1 \in \mathbb{Z}$. If $b \in \mathbb{N}$, then by 3.1 and 3.2 we conclude that

$$\begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \varepsilon^{l(a_1 l - a_1 b_0 - a_1 b_1 l - a_0 b_1)} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \binom{b_1 + p^{h-1}}{a_1}$$

$$= \varepsilon^{-alp^{h-1}} \varepsilon^{l(a_1 l - a_1 b_0 - a_1 b_1 l - a_0 b_1)} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \binom{b_1}{a_1}$$

$$= \varepsilon^{-alp^{h-1}} \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon.$$

Furthermore if $b + lp^{h-1} < 0$, then $-b + a - 1 - lp^{h-1} \geqslant 0$ and hence

$$\begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon = (-1)^a \begin{bmatrix} -b + a - 1 - lp^{h-1} \\ a \end{bmatrix}_\varepsilon = (-1)^a \varepsilon^{alp^{h-1}} \begin{bmatrix} -b + a - 1 \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon.$$

Now we assume $-lp^{h-1} \leqslant b < 0$. According to 3.1 we have

$$(3.3.1) \qquad \begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \varepsilon^{l(a_1 l - a_1 b_0 - ab_1)} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \binom{b_1}{a_1}.$$

If $a_0 - b_0 - 1 \geqslant 0$ then $\begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon = (-1)^{a_0} \begin{bmatrix} a_0 - b_0 - 1 \\ a_0 \end{bmatrix}_\varepsilon = 0$ and hence, by 3.1 and (3.3.1), we have

$$\begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon = (-1)^a \begin{bmatrix} l(a_1 - b_1) + (a_0 - b_0 - 1) \\ a \end{bmatrix}_\varepsilon$$

$$= (-1)^a \varepsilon^{l(a_1 l - a_1(a_0 - b_0 - 1) - a(a_1 - b_1))} \begin{bmatrix} a_0 - b_0 - 1 \\ a_0 \end{bmatrix}_\varepsilon \binom{a_1 - b_1}{a_1}$$

$$= 0$$

$$= \varepsilon^{alp^{h-1}} \begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon.$$

Now we assume $-lp^{h-1} \leqslant b < 0$ and $a_0 - b_0 - 1 < 0$. Then $a_1 - b_1 - 1 \geqslant 0$ and $0 \leqslant l + a_0 - b_0 - 1 < l$. According to 3.1 we have

$$
\begin{aligned}
\begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon &= (-1)^a \begin{bmatrix} -b + a - 1 \\ a \end{bmatrix}_\varepsilon \\
&= (-1)^a \begin{bmatrix} l(a_1 - b_1 - 1) + (l + a_0 - b_0 - 1) \\ a \end{bmatrix}_\varepsilon \\
&= (-1)^a \varepsilon^{l(-a_1(a_0 - b_0 - 1) - a(a_1 - b_1 - 1))} \begin{bmatrix} l + a_0 - b_0 - 1 \\ a_0 \end{bmatrix}_\varepsilon \begin{pmatrix} a_1 - b_1 - 1 \\ a_1 \end{pmatrix} \\
&= (-1)^{a_1 l + a_1} \varepsilon^{l(-a_1(a_0 - b_0 - 1) - a(a_1 - b_1 - 1))} \begin{bmatrix} b_0 - l \\ a_0 \end{bmatrix}_\varepsilon \begin{pmatrix} b_1 \\ a_1 \end{pmatrix}.
\end{aligned}
$$

Since $0 \leqslant a_0 < l$ and $[m + l]_\varepsilon = \varepsilon^{-l}[m]_\varepsilon$ we see that $\begin{bmatrix} b_0 - l \\ a_0 \end{bmatrix} = \varepsilon^{a_0 l} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon$. This implies that

(3.3.2)
$$
\begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon = (-1)^{a_1 l + a_1} \varepsilon^{l(a_0 - a_1(a_0 - b_0 - 1) - a(a_1 - b_1 - 1))} \begin{bmatrix} b_0 \\ a_0 \end{bmatrix}_\varepsilon \begin{pmatrix} b_1 \\ a_1 \end{pmatrix}.
$$

Furthermore since $\varepsilon^{2l} = 1$ and $(a_1^2 l - a_1) - (a_1 l + a_1) = -2a_1 + la_1(a_1 - 1)$ is even, we see that

$$
\begin{aligned}
\frac{\varepsilon^{l(a_1 l - a_1 b_0 - a b_1)}}{\varepsilon^{l(a_0 - a_1(a_0 - b_0 - 1) - a(a_1 - b_1 - 1))}} &= \varepsilon^{l(-2ab_1 - 2a_1 b_0 - 2a_0 + 2a_0 a_1)} \varepsilon^{l(a_1^2 l - a_1)} \\
&= \varepsilon^{l(a_1^2 l - a_1)} = \varepsilon^{l(a_1 l + a_1)} = (-1)^{a_1(l+1)}.
\end{aligned}
$$

Thus by (3.3.1) and (3.3.2) we conclude that $\begin{bmatrix} b + lp^{h-1} \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \begin{bmatrix} b \\ a \end{bmatrix}_\varepsilon$. The proof is completed.
$\square$

Corollary 3.4. *Assume* $0 \leqslant a, b < lp^{h-1}$ *and* $a + b \geqslant lp^{h-1}$. *Then* $\begin{bmatrix} a+b \\ a \end{bmatrix}_\varepsilon = 0$.

Proof. According to 3.3 we have $\begin{bmatrix} a+b \\ a \end{bmatrix}_\varepsilon = \varepsilon^{-alp^{h-1}} \begin{bmatrix} a+b-lp^{h-1} \\ a \end{bmatrix}_\varepsilon$. Since $0 \leqslant a + b - lp^{h-1} < a$, we see that $\begin{bmatrix} a+b-lp^{h-1} \\ a \end{bmatrix}_\varepsilon = 0$. The assertion follows.
$\square$

Let $\widetilde{\mathfrak{u}}_{\Bbbk}^0(n)_h$ be the $\Bbbk$-subalgebra of $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h$ generated by $K_j^{\pm 1}$, $\begin{bmatrix} K_j; 0 \\ t \end{bmatrix}$ for $1 \leqslant j \leqslant n$ and $0 \leqslant t < lp^{h-1}$. For $h \geqslant 1$ let

$$
\mathbb{N}_{lp^{h-1}}^n = \{ \lambda \in \mathbb{N}^n \mid 0 \leqslant \lambda_i < lp^{h-1}, \forall i \}.
$$

Lemma 3.5. *The set* $\mathfrak{M}^0 = \{ \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix} \mid \delta \in \mathbb{N}^n, \delta_i \in \{0, 1\}, \lambda \in \mathbb{N}_{lp^{h-1}}^n \}$ *forms a* $\Bbbk$*-basis for* $\widetilde{\mathfrak{u}}_{\Bbbk}^0(n)_h$.

Proof. Let $V_1 = \text{span}_{\Bbbk} \mathfrak{M}^0$. From 2.1, we see that the set $\mathfrak{M}^0$ is linearly independent. Thus it is enough to prove that $\widetilde{\mathfrak{u}}_{\Bbbk}^0(n)_h = V_1$. Let V_2 be the $\Bbbk$-submodule of $\widetilde{\mathfrak{u}}_{\Bbbk}^0(n)_h$ spanned by the elements $\prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix}$ ($\delta \in \mathbb{Z}^n$, $\lambda \in \mathbb{N}^n$, $0 \leqslant \lambda_i < lp^{h-1}$, for all i). According to [17, 2.3(g8)], for $0 \leqslant t, t' < lp^{h-1}$ we have

$$
\varepsilon^{t't} \begin{bmatrix} K_i; 0 \\ t' \end{bmatrix} \begin{bmatrix} K_i; 0 \\ t \end{bmatrix} = \begin{bmatrix} t + t' \\ t \end{bmatrix}_\varepsilon \begin{bmatrix} K_i; 0 \\ t + t' \end{bmatrix} - \sum_{0 < j \leqslant t'} (-1)^j \varepsilon^{t(t'-j)} \begin{bmatrix} t + j - 1 \\ j \end{bmatrix}_\varepsilon K_i^j \begin{bmatrix} K_i; 0 \\ t' - j \end{bmatrix} \begin{bmatrix} K_i; 0 \\ t \end{bmatrix}.
$$

Note that by 3.4 we have $\left[\begin{smallmatrix} t+t' \\ t \end{smallmatrix}\right]_\varepsilon \left[\begin{smallmatrix} K_i;0 \\ t+t' \end{smallmatrix}\right] = 0$ for $0 \leqslant t, t' < lp^{h-1}$ with $t + t' \geqslant lp^{h-1}$. Thus, by induction on t' we see that $\left[\begin{smallmatrix} K_i;0 \\ t' \end{smallmatrix}\right]\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right] \in V_2$ for $0 \leqslant t, t' < lp^{h-1}$. It follows that $\widetilde{\mathfrak{u}}^0_k(n)_h = V_2$. Furthermore, by the proof of [17, 2.14], for $m \geqslant 0$ and $0 \leqslant t < lp^{h-1}$ we have

$$K_i^{m+2} \begin{bmatrix} K_i;0 \\ t \end{bmatrix} = \varepsilon^t(\varepsilon^{t+1} - \varepsilon^{-t-1})K_i^{m+1} \begin{bmatrix} K_i;0 \\ t+1 \end{bmatrix} + \varepsilon^{2t}K_i^m \begin{bmatrix} K_i;0 \\ t \end{bmatrix},$$

$$K_i^{-m-1} \begin{bmatrix} K_i;0 \\ t \end{bmatrix} = -\varepsilon^{-t}(\varepsilon^{t+1} - \varepsilon^{-t-1})K_i^{-m} \begin{bmatrix} K_i;0 \\ t+1 \end{bmatrix} + \varepsilon^{-2t}K_i^{-m+1} \begin{bmatrix} K_i;0 \\ t \end{bmatrix}.$$

If $t+1 = lp^{h-1}$, then $\varepsilon^t(\varepsilon^{t+1} - \varepsilon^{-t-1})K_i^{m+1}\left[\begin{smallmatrix} K_i;0 \\ t+1 \end{smallmatrix}\right] = -\varepsilon^{-t}(\varepsilon^{t+1} - \varepsilon^{-t-1})K_i^{-m}\left[\begin{smallmatrix} K_i;0 \\ t+1 \end{smallmatrix}\right] = 0$. Thus by induction on $m \geqslant 0$ we see that $K_i^{\pm m}\left[\begin{smallmatrix} K_i;0 \\ t \end{smallmatrix}\right] \in V_1$ for $0 \leqslant t < lp^{h-1}$. This implies that $V_1 = V_2$. The assertion follows. $\square$

We are now ready to prove 3.7. Let $\Theta^\pm(n)_h = \{A \in \Theta^\pm(n) \mid 0 \leqslant a_{s,t} < lp^{h-1}, \forall s \neq t\}$.

Lemma 3.6. *The algebra $\widetilde{\mathfrak{u}}_k(n)_h$ is generated as a k-module by the elements $A(\delta, \lambda)$ for $A \in \Theta^\pm(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$.*

Proof. Let V_h be the k-submodule of $U_k(n)$ spanned by $A(\delta, \lambda)$ for $A \in \Theta^\pm(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$. According to [15, 3.5(1)] for $A \in \Theta^\pm(n)_h$, $0 \leqslant m < lp^{h-1}$, $1 \leqslant i \leqslant n-1$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$, we have

$$(mE_{i,i+1})(\mathbf{0})A(\delta, \lambda)$$

$$= \sum_{\substack{\mathbf{t} \in \Lambda(n,m),\, 0 \leqslant j \leqslant \lambda_i \\ t_u \leqslant a_{i+1,u},\, \forall u \neq i+1 \\ 0 \leqslant k \leqslant \lambda_{i+1},\, 0 \leqslant c \leqslant \min\{t_i, j\}}} f^{\mathbf{t}}_{j,c,k}\left(A + \sum_{u \neq i} t_u E_{i,u} - \sum_{u \neq i+1} t_u E_{i+1,u}\right)(\delta + \alpha^{\mathbf{t}}_{j,c,k}, \lambda + \beta^{\mathbf{t}}_{j,c,k}).$$

where $\alpha^{\mathbf{t}}_{j,c,k} = \left(\sum_{i>u} t_u + \lambda_i - j - c\right)e_i + \left(\lambda_{i+1} - k - \sum_{i+1>u} t_u\right)e_{i+1}$, $\beta^{\mathbf{t}}_{j,c,k} = (t_i + j - c - \lambda_i)e_i + (k - \lambda_{i+1})e_{i+1}$ with $e_i = (0, \cdots, 0, \underset{i}{1}, 0 \cdots, 0) \in \mathbb{N}^n$, and

$$f^{\mathbf{t}}_{j,c,k} = \varepsilon^{g^{\mathbf{t}}_{j,k}} \prod_{u \neq i} \begin{bmatrix} a_{i,u} + t_u \\ t_u \end{bmatrix}_\varepsilon \begin{bmatrix} -t_i \\ \lambda_i - j \end{bmatrix}_\varepsilon \begin{bmatrix} t_i + j - c \\ t_i \end{bmatrix}_\varepsilon \begin{bmatrix} t_i \\ c \end{bmatrix}_\varepsilon \begin{bmatrix} t_{i+1} \\ \lambda_{i+1} - k \end{bmatrix}_\varepsilon$$

with $g^{\mathbf{t}}_{j,k} = \sum_{j>u, j \neq i} a_{i,j}t_u - \sum_{j>u, j \neq i+1} a_{i+1,j}t_u + \sum_{u' \neq i,i+1,\, u < u'} t_u t_{u'} - t_i \delta_i + t_{i+1}\delta_{i+1} + 2jt_i - kt_{i+1}$. If $A + \sum_{u \neq i} t_u E_{i,u} - \sum_{u \neq i+1} t_u E_{i+1,u} \notin \Theta^\pm(n)_h$ then $a_{i,u} + t_u \geqslant lp^{h-1}$ for some $u \neq i$. From 3.4 we see that $\left[\begin{smallmatrix} a_{i,u}+t_u \\ t_u \end{smallmatrix}\right]_\varepsilon = 0$ and hence $f^{\mathbf{t}}_{j,c,k} = 0$. Furthermore, if $\lambda + \beta^{\mathbf{t}}_{j,c,k} \notin \mathbb{N}^n_{lp^{h-1}}$ then $(\lambda + \beta^t_{j,c,k})_i = t_i + j - c \geqslant lp^{h-1}$. From 3.4 we see that $\left[\begin{smallmatrix} t_i+j-c \\ t_i \end{smallmatrix}\right]_\varepsilon = 0$ and hence $f^{\mathbf{t}}_{j,c,k} = 0$. Thus we conclude that

(3.6.1) $$(mE_{i,i+1})(\mathbf{0})V_h \subseteq V_h,$$

for $0 \leqslant m < lp^{h-1}$ and $1 \leqslant i \leqslant n-1$. Similarly, using [15, 3.4,3.5(2)] we see that

(3.6.2) $$(mE_{i+1,i})(\mathbf{0})V_h \subseteq V_h \text{ and } 0(\gamma, \mu)V_h \subseteq V_h$$

for $0 \leqslant m < lp^{h-1}$, $1 \leqslant i \leqslant n-1$, $\gamma \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}^n_{lp^{h-1}}$. Combining (3.6.1) with (3.6.2) implies that

$$(3.6.3) \qquad\qquad \widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h \subseteq \widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h V_h \subseteq V_h.$$

On the other hand, from [15, 3.4] we see that for $A \in \Theta^{\pm}(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$,

$$(3.6.4) \qquad A(\mathbf{0})0(\delta, \lambda) = \varepsilon^{\mathrm{co}(A)\cdot(\delta+\lambda)} A(\delta, \lambda) + \sum_{\mathbf{j} \in \mathbb{N}^n, \mathbf{0} < \mathbf{j} \leqslant \lambda} \varepsilon^{\mathrm{co}(A)\cdot(\delta+\lambda-\mathbf{j})} \begin{bmatrix} \mathrm{co}(A) \\ \mathbf{j} \end{bmatrix} A(\delta - \mathbf{j}, \lambda - \mathbf{j}).$$

This implies that

$$(3.6.5) \qquad\qquad V_h = \mathrm{span}_{\mathcal{k}}\{A(\mathbf{0})0(\delta, \lambda) \mid A \in \Theta^{\pm}(n)_h, \delta \in \mathbb{Z}^n, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}.$$

Furthermore, combining (2.3.1) with 2.3 shows that for $A \in \Theta^{\pm}(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$,

$$E^{(A^+)} F^{(A^-)} \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix} = E^{(A^+)} F^{(A^-)} 0(\delta, \lambda) = A(\mathbf{0})0(\delta, \lambda) + f$$

where f is a $\mathcal{k}$-linear combination of $B(\mathbf{0})0(\gamma, \mu)$ with $B \in \Theta^{\pm}(n)$, $B \prec A$, $\gamma \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}^n$. From (3.6.3) and (3.6.5) we see that f must be a $\mathcal{k}$-linear combination of $B(\mathbf{0})0(\gamma, \mu)$ with $B \in \Theta^{\pm}(n)_h$, $B \prec A$, $\gamma \in \mathbb{Z}^n$ and $\mu \in \mathbb{N}^n_{lp^{h-1}}$. Thus we conclude that

$(3.6.6)$
$$V_h = \mathrm{span}_{\mathcal{k}} \left\{ E^{(A^+)} F^{(A^-)} \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix} \mid A \in \Theta^{\pm}(n)_h, \delta \in \mathbb{Z}^n, \lambda \in \mathbb{N}^n_{lp^{h-1}} \right\} \subseteq \widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h.$$

The assertion follows. $\square$

Proposition 3.7. *Each of the following set forms a $\mathcal{k}$-basis for $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h$:*

(1) $\mathfrak{M} := \{E^{(A^+)} \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix} F^{(A^-)} \mid A \in \Theta^{\pm}(n)_h, \delta \in \mathbb{N}^n, \delta_i \in \{0, 1\}, \forall i, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$;

(2) $\mathfrak{B} := \{A(\delta, \lambda) \mid A \in \Theta^{\pm}(n)_h, \delta \in \mathbb{N}^n, \delta_i \in \{0, 1\}, \forall i, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$;

(3) $\mathfrak{B}' := \{A(\mathbf{0})0(\delta, \lambda) \mid A \in \Theta^{\pm}(n)_h, \delta \in \mathbb{N}^n, \delta_i \in \{0, 1\}, \forall i, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$.

Proof. According to 2.1 and 2.3, it is enough to prove that $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h = \mathrm{span}_{\mathcal{k}} \mathfrak{M} = \mathrm{span}_{\mathcal{k}} \mathfrak{B} = \mathrm{span}_{\mathcal{k}} \mathfrak{B}'$. From 3.5, 3.6, (3.6.5) and (3.6.6) we see that $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h = \mathrm{span}_{\mathcal{k}} \mathfrak{M} = \mathrm{span}_{\mathcal{k}} \mathfrak{B}'$. For $A \in \Theta^{\pm}(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ we have

$$A(\delta, \lambda) = \varepsilon^{\lambda_i}(\varepsilon^{\lambda_i+1} - v^{-\lambda_i-1}) A(\delta - \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) + \varepsilon^{2\lambda_i} A(\delta - 2\boldsymbol{e}_i, \lambda)$$

$$= -\varepsilon^{-\lambda_i}(\varepsilon^{\lambda_i+1} - \varepsilon^{-\lambda_i-1}) A(\delta + \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) + \varepsilon^{-2\lambda_i} A(\delta + 2\boldsymbol{e}_i, \lambda)$$

Note that if $\lambda_i + 1 = lp^{h-1}$ then $\varepsilon^{\lambda_i}(\varepsilon^{\lambda_i+1} - v^{-\lambda_i-1}) A(\delta - \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) = -\varepsilon^{-\lambda_i}(\varepsilon^{\lambda_i+1} - \varepsilon^{-\lambda_i-1}) A(\delta + \boldsymbol{e}_i, \lambda + \boldsymbol{e}_i) = 0$. This together with 3.6 shows that $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h = \mathrm{span}_{\mathcal{k}} \mathfrak{B}$. $\square$

4. The algebra $\mathscr{K}'(n)_h$

We will construct the algebra $\mathscr{K}'(n)_h$ in this section. We will prove in 5.5 the algebra $\mathscr{K}'(n)_h$ is the realization of $\widetilde{\mathfrak{u}}_{\Bbbk}(n)_h$.

Let $\mathcal{K}_{\Bbbk}(n) = \mathcal{K}_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} \Bbbk$, where $\Bbbk$ is regarded as a $\mathcal{Z}$-module by specializing v to ε. For $A \in \widetilde{\Theta}(n)$ let

$$[A]_\varepsilon = [A] \otimes 1 \in \mathcal{K}_{\Bbbk}(n).$$

Let $\widetilde{\Theta}(n)_h$ be the set of all $A = (a_{i,j}) \in \widetilde{\Theta}(n)$ such that $a_{i,j} < lp^{h-1}$ for all $i \neq j$. We will denote by $\mathscr{K}(n)_h$ the $\Bbbk$-submodule of $\mathcal{K}_{\Bbbk}(n)$ spanned by the elements $[A]_\varepsilon$ with $A \in \widetilde{\Theta}(n)_h$.

To construct the algebra $\mathscr{K}'(n)_h$ we need the following lemma (cf. [1, 6.2] and [14, 5.1]).

Lemma 4.1. (1) $\mathscr{K}(n)_h$ is a subalgebra of $\mathcal{K}_{\Bbbk}(n)$. It is generated by $[mE_{h,h+1} + \mathrm{diag}(\lambda)]_\varepsilon$ and $[mE_{h+1,h} + \mathrm{diag}(\lambda)]_\varepsilon$ for $0 \leqslant m < lp^{h-1}$, $1 \leqslant h \leqslant n-1$ and $\lambda \in \mathbb{Z}^n$.

(2) Let D be any diagonal matrix in $\widetilde{\Theta}(n)$. The map $\tau_D : \mathscr{K}(n)_h \to \mathscr{K}(n)_h$ given by $[A]_\varepsilon \to [A + l'p^{h-1}D]_\varepsilon$ is an algebra homomorphism.

Proof. Let $A = (a_{s,t}) \in \widetilde{\Theta}(n)_h$ and $0 \leqslant m < lp^{h-1}$. Assume that $B = (b_{s,t}) \in \widetilde{\Theta}(n)_h$ is such that $B - mE_{i,i+1}$ is a diagonal matrix such that $\mathrm{co}(B) = \mathrm{ro}(A)$. According to [1, 4.6(a)] we have

$$[B]_\varepsilon \cdot [A]_\varepsilon = \sum_{\substack{\mathbf{t} \in \Lambda(n,m) \\ \forall u \neq i+1, t_u \leqslant a_{i+1,u}}} \varepsilon^{\beta(\mathbf{t},A)} \prod_{1 \leqslant u \leqslant n} \begin{bmatrix} a_{i,u} + t_u \\ t_u \end{bmatrix}_\varepsilon \left[A + \sum_{1 \leqslant u \leqslant n} t_u(E_{i,u} - E_{i+1,u}) \right]_\varepsilon$$

where $\beta(\mathbf{t}, A) = \sum_{j>u} a_{i,j}t_u - \sum_{j>u} a_{i+1,j}t_u + \sum_{u<u'} t_u t_{u'}$. Assume that $A + \sum_u t_u(E_{i,u} - E_{i+1,u}) \notin \widetilde{\Theta}(n)_h$ for some $\mathbf{t}$; then $a_{i,u} + t_u \geqslant lp^{h-1}$ for some $u \neq i$. Since $0 \leqslant a_{i,u}, t_u < lp^{h-1}$, by 3.4, we conclude that $\begin{bmatrix} a_{i,u} + t_u \\ t_u \end{bmatrix}_\varepsilon = 0$ and hence $[B]_\varepsilon \cdot [A]_\varepsilon \in \mathscr{K}(n)_h$. Similarly, we have $[C]_\varepsilon \cdot [A]_\varepsilon \in \mathscr{K}(n)_h$, where C is such that $C - mE_{i+1,i}$ is a diagonal matrix such that $\mathrm{co}(C) = \mathrm{ro}(A)$. Now using [1, 4.6(c)], (1) can be proved in a way similar to the proof of [1, 6.2].

According to [1, 4.6(a),(b)] and 3.3 we see that $\tau_D([A']_\varepsilon [A]_\varepsilon) = \tau_D([A']_\varepsilon)\tau_D([A]_\varepsilon)$ for any A' of the form B, C as above. Since $\mathscr{K}(n)_h$ is generated by elements like $[B]_\varepsilon$, $[C]_\varepsilon$ above, we conclude that τ_D is an algebra homomorphism. $\square$

Let $\widetilde{\Theta}'(n)_h$ be the set of all $n \times n$ matrices $A = (a_{i,j})$ with $a_{i,j} \in \mathbb{N}$, $a_{i,j} < lp^{h-1}$ for all $i \neq j$ and $a_{i,i} \in \mathbb{Z}/l'p^{h-1}\mathbb{Z}$ for all i. We have an obvious map $pr : \widetilde{\Theta}(n)_h \to \widetilde{\Theta}'(n)_h$ defined by reducing the diagonal entries modulo $l'p^{h-1}\mathbb{Z}$.

Let $\mathscr{K}'(n)_h$ be the free $\Bbbk$-module with basis $\{[A]_\varepsilon \mid A \in \widetilde{\Theta}'(n)_h\}$. We shall define an algebra structure on $\mathscr{K}'(n)_h$ as follows. If the column sums of A are not equal to the row sums of A' (as integers modulo $l'p^{h-1}$), then the product $[A]_\varepsilon \cdot [A']_\varepsilon$ for $A, A' \in \widetilde{\Theta}'(n)_h$ is zero. Assume now that the column sums of A are equal to the row sums of A' (as integers modulo $l'p^{h-1}$). We can then represent A, A' by elements $\widetilde{A}, \widetilde{A}' \in \widetilde{\Theta}(n)_h$ such that the column sums of $\widetilde{A}$ are equal to the row sums of $\widetilde{A}'$ (as integers). According to 4.1(1), we can write $[\widetilde{A}]_\varepsilon \cdot [\widetilde{A}']_\varepsilon = \sum_{\widetilde{A}'' \in I} \rho_{\widetilde{A}''}[\widetilde{A}'']_\varepsilon$

(product in $\mathscr{K}(n)_h$) where $I = \{\widetilde{A}'' \in \widetilde{\Theta}(n)_h \mid \mathrm{ro}(\widetilde{A}'') = \mathrm{ro}(\widetilde{A}), \ \mathrm{co}(\widetilde{A}'') = \mathrm{co}(\widetilde{A}')\}$ (a finite set) and $\rho_{\widetilde{A}''} \in \mathcal{k}$. Then the product $[A]_\varepsilon \cdot [A']_\varepsilon$ is defined to be $\sum_{\widetilde{A}'' \in I} \rho_{\widetilde{A}''} [pr(\widetilde{A}'')]_\varepsilon$. From 4.1(2) we see that the product is well defined and $\mathscr{K}'(n)_h$ becomes an associative algebra over $\mathcal{k}$.

In the case where l' is odd, the algebra $\mathscr{K}'(n)_1$ is the algebra $\mathscr{K}'$ constructed in [1, 6.3]. Furthermore, it was remarked at the end of [1] that $\mathscr{K}'$ is "essentially" the algebra defined in [17, §5] for type A. We will prove in 5.5 that $\mathscr{K}'(n)_h$ is isomorphic to the algebra $\mathfrak{u}_{\mathcal{k}}(n)_h$ in the case where l' is odd.

Mimicking the construction of $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$, we define $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ to be the $\mathcal{k}$-module of all formal $\mathcal{k}$-linear combinations $\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]_\varepsilon$ satisfying the property (2.1.1). The product of two elements $\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]_\varepsilon, \ \sum_{B \in \widetilde{\Theta}(n)} \gamma_B [B]_\varepsilon$ in $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ is defined to be $\sum_{A,B} \beta_A \gamma_B [A]_\varepsilon \cdot [B]_\varepsilon$ where $[A]_\varepsilon \cdot [B]_\varepsilon$ is the product in $\mathcal{K}_{\mathcal{k}}(n)$. Then $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ becomes an associative algebra over $\mathcal{k}$.

We end this section by interpreting $\mathscr{K}'(n)_h$ as a $\mathcal{k}$-subalgebra of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$. For $h \geqslant 1$ let $\mathbb{Z}_{l'p^{h-1}} = \mathbb{Z}/l'p^{h-1}\mathbb{Z}$ and let $^- : \mathbb{Z}^n \to (\mathbb{Z}_{l'p^{h-1}})^n$ be the map defined by $\overline{(j_1, j_2, \cdots, j_n)} = (\overline{j_1}, \overline{j_2}, \cdots, \overline{j_n})$. For $A \in \Theta^\pm(n)_h$ and $\bar{\mu} \in (\mathbb{Z}_{l'p^{h-1}})^n$ let

$$(4.1.1) \qquad\qquad [\![A + \mathrm{diag}(\bar{\mu})]\!]_h = \sum_{\substack{\nu \in \mathbb{Z}^n \\ \bar{\mu} = \bar{\nu}}} [A + \mathrm{diag}(\nu)]_\varepsilon.$$

Let $\mathcal{W}_{\mathcal{k}}(n)_h$ be the $\mathcal{k}$-submodule of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ spanned by the set $\{[\![A + \mathrm{diag}(\bar{\lambda})]\!]_h \mid A \in \Theta^\pm(n)_h, \ \bar{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n\}$. From 4.1 we see that $\mathcal{W}_{\mathcal{k}}(n)_h$ is a $\mathcal{k}$-subalgebra of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$. Furthermore, it is easy to prove that there is an algebra isomorphism

$$(4.1.2) \qquad\qquad \mathcal{W}_{\mathcal{k}}(n)_h \xrightarrow{\sim} \mathscr{K}'(n)_h$$

defined by sending $[\![A]\!]_h$ to $[A]_\varepsilon$ for $A \in \widetilde{\Theta}'(n)_h$.

5. Realization of $\mathfrak{u}_{\mathcal{k}}(n)_h$

For $A \in \Theta^\pm(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$ let

$$A(\delta, \lambda)_\varepsilon = \sum_{\mu \in \mathbb{Z}^n} \varepsilon^{\mu \cdot \delta} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_\varepsilon [A + \mathrm{diag}(\mu)]_\varepsilon \in \widehat{\mathcal{K}}_{\mathcal{k}}(n).$$

Let $\mathcal{V}_{\mathcal{k}}(n)$ be the $\mathcal{k}$-submodule of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ spanned by the elements $A(\delta, \lambda)_\varepsilon$ for $A \in \Theta^\pm(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$. For $h \geqslant 1$ let $\mathcal{V}_{\mathcal{k}}(n)_h$ be the $\mathcal{k}$-submodule of $\widehat{\mathcal{K}}_{\mathcal{k}}(n)$ spanned by the elements $A(\delta, \lambda)_\varepsilon$ for $A \in \Theta^\pm(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$. We will prove in 5.5 that $\mathfrak{u}_{\mathcal{k}}(n)_h \cong \mathcal{V}_{\mathcal{k}}(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is odd, and that $\widetilde{\mathfrak{u}}_{\mathcal{k}}(n)_h \cong \mathcal{V}_{\mathcal{k}}(n)_h \cong \mathscr{K}'(n)_h$ in the case where l' is even and $\mathcal{k}$ is a field.

Let $\widehat{\mathcal{K}}_{\mathcal{Z}}(n)$ be the $\mathcal{Z}$-submodule of $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ consisting of the elements $\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]$ with $\beta_A \in \mathcal{Z}$. Then $\widehat{\mathcal{K}}_{\mathcal{Z}}(n)$ is a $\mathcal{Z}$-subalgebra of $\widehat{\mathcal{K}}_{\mathcal{Q}}(n)$. There is a natural algebra homomorphism

$$\theta : \widehat{\mathcal{K}}_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} \mathcal{k} \to \widehat{\mathcal{K}}_{\mathcal{k}}(n)$$

defined by sending $(\sum_{A \in \widetilde{\Theta}(n)} \beta_A[A]) \otimes 1$ to $\sum_{A \in \widetilde{\Theta}(n)} (\beta_A \cdot 1)[A]_\varepsilon$, where 1 is the identity element in $\mathfrak{k}$.

Recall the injective algebra homomorphism $\varphi : U_{\mathcal{Q}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Q}}(n)$ defined in 2.2. From 2.3 we see that $\varphi(U_{\mathcal{Z}}(n)) \subseteq \widehat{\mathcal{K}}_{\mathcal{Z}}(n)$. Thus, by restriction, we get a map $\varphi : U_{\mathcal{Z}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Z}}(n)$. It induces an algebra homomorphism $\varphi_{\mathfrak{k}} : U_{\mathfrak{k}}(n) \to \widehat{\mathcal{K}}_{\mathcal{Z}}(n) \otimes_{\mathcal{Z}} \mathfrak{k}$. The map θ, composed with $\varphi_{\mathfrak{k}}$ gives an algebra homomorphism

$$(5.0.3) \qquad \xi := \theta \circ \varphi_{\mathfrak{k}} : U_{\mathfrak{k}}(n) \to \widehat{\mathcal{K}}_{\mathfrak{k}}(n).$$

By definition we have $\xi(A(\delta, \lambda)) = A(\delta, \lambda)_\varepsilon$ for $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n$. This together with 2.3 and 3.6 implies that

$$(5.0.4) \qquad \xi(U_{\mathfrak{k}}(n)) = \mathcal{V}_{\mathfrak{k}}(n) \text{ and } \xi(\widetilde{\mathfrak{u}}_{\mathfrak{k}}(n)_h) = \mathcal{V}_{\mathfrak{k}}(n)_h.$$

In particular, $\mathcal{V}_{\mathfrak{k}}(n)$ and $\mathcal{V}_{\mathfrak{k}}(n)_h$ are all $\mathfrak{k}$-subalgebras of $\widehat{\mathcal{K}}_{\mathfrak{k}}(n)$.

We will now construct several bases for $\mathcal{V}_{\mathfrak{k}}(n)_h$ and $\mathcal{V}_{\mathfrak{k}}(n)$ in 5.1 and 5.3. These results will be used to prove 5.5. According to 3.3 we see that $\begin{bmatrix} \nu \\ \lambda \end{bmatrix}_\varepsilon = \begin{bmatrix} \nu + l'p^{h-1}\delta \\ \lambda \end{bmatrix}_\varepsilon$ for $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ and $\nu, \delta \in \mathbb{Z}^n$. This implies that

$$(5.0.5) \qquad A(\delta, \lambda)_\varepsilon = \sum_{\bar{\mu} \in (\mathbb{Z}_{l'p^{h-1}})^n} \varepsilon^{\delta \cdot \mu} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_\varepsilon \llbracket A + \operatorname{diag}(\bar{\mu}) \rrbracket_h$$

for $A \in \Theta^{\pm}(n)_h$, $\delta \in \mathbb{Z}^n$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$, where $\llbracket A + \operatorname{diag}(\bar{\mu}) \rrbracket_h$ is defined in (4.1.1). For $\lambda, \mu \in \mathbb{N}^n$, we write $\lambda \leqslant \mu$ if and only if $\lambda_i \leqslant \mu_i$ for $1 \leqslant i \leqslant n$. If $\lambda \leqslant \mu$ and $\lambda_i < \mu_i$ for some $1 \leqslant i \leqslant n$ then we write $\lambda < \mu$.

Lemma 5.1. *Assume l' is odd. Then $\mathcal{V}_{\mathfrak{k}}(n)_h = \mathcal{W}_{\mathfrak{k}}(n)_h$ and the set $\mathcal{N}_h := \{A(\mathbf{0}, \lambda)_\varepsilon \mid A \in \Theta^{\pm}(n)_h, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$ forms a $\mathfrak{k}$-basis for $\mathcal{V}_{\mathfrak{k}}(n)_h$. Furthermore, if $p > 0$, then the set $\mathcal{N} := \{A(\mathbf{0}, \lambda) \mid A \in \Theta^{\pm}(n), \lambda \in \mathbb{N}^n\}$ forms a $\mathfrak{k}$-basis for $\mathcal{V}_{\mathfrak{k}}(n)$.*

Proof. From (5.0.5) we see that for $A \in \Theta^{\pm}(n)_h$ and $\lambda \in \mathbb{N}^n_{lp^{h-1}}$,

$$A(\mathbf{0}, \lambda)_\varepsilon = \llbracket A + \operatorname{diag}(\bar{\lambda}) \rrbracket_h + \sum_{\mu \in \mathbb{N}^n_{lp^{h-1}}, \lambda < \mu} \begin{bmatrix} \mu \\ \lambda \end{bmatrix}_\varepsilon \llbracket A + \operatorname{diag}(\bar{\mu}) \rrbracket_h.$$

This, together with the fact that the set $\mathcal{L}_h$ forms a $\mathfrak{k}$-basis for $\mathcal{W}_{\mathfrak{k}}(n)_h$, shows that the set $\mathcal{N}_h$ forms a $\mathfrak{k}$-basis for $\mathcal{W}_{\mathfrak{k}}(n)_h$. It follows that $\mathcal{W}_{\mathfrak{k}}(n)_h \subseteq \mathcal{V}_{\mathfrak{k}}(n)_h$. Furthermore from (5.0.5) we see that $\mathcal{V}_{\mathfrak{k}}(n)_h \subseteq \mathcal{W}_{\mathfrak{k}}(n)_h$. Thus $\mathcal{V}_{\mathfrak{k}}(n)_h = \mathcal{W}_{\mathfrak{k}}(n)_h$. Now we assume $p = \operatorname{char} \mathfrak{k} > 0$. Since $\mathcal{V}_{\mathfrak{k}}(n) = \bigcup_{h \geqslant 1} \mathcal{V}_{\mathfrak{k}}(n)_h$, $\mathcal{N} = \bigcup_{h \geqslant 1} \mathcal{N}_h$ and the set $\mathcal{N}_h$ forms a $\mathfrak{k}$-basis for $\mathcal{V}_{\mathfrak{k}}(n)_h$, we conclude that the set $\mathcal{N}$ forms a $\mathfrak{k}$-basis for $\mathcal{V}_{\mathfrak{k}}(n)$. $\qquad \square$

Lemma 5.2. *For $m \geqslant 1$, let $X_m = ((-1)^{\delta \cdot \beta})_{\delta, \beta \in \mathcal{I}_m}$, where $\mathcal{I}_m = \{\delta \in \mathbb{N}^m \mid \delta_i \in \{0, 1\} \text{ for } 1 \leqslant i \leqslant m\}$. If we order the set $\mathcal{I}_m$ lexicographically, then $\det(X_m) = (-2)^m$ for all m.*

Proof. Since $\mathcal{I}_m = \{(0, \delta) \mid \delta \in \mathcal{I}_{m-1}\} \cup \{(1, \delta) \mid \delta \in \mathcal{I}_{m-1}\}$ we see that

$$X_m = \begin{pmatrix} X_{m-1} & X_{m-1} \\ X_{m-1} & -X_{m-1} \end{pmatrix}.$$

This, together with the fact that $\det(X_1) = -2$, implies that

$$\det(X_m) = \det \begin{pmatrix} X_{m-1} & X_{m-1} \\ 0 & -2X_{m-1} \end{pmatrix} = -2\det(X_{m-1})^2 = (-2)^{2^m-1}$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 5.3. *Assume l' is even and $\Bbbk$ is a field. Then $\mathcal{V}_{\Bbbk}(n)_h = \mathcal{W}_{\Bbbk}(n)_h$ and the set $\mathcal{B}_h :=$ $\{A(\delta, \lambda)_\varepsilon \mid A \in \Theta^{\pm}(n)_h, \lambda \in \mathbb{N}^n_{lp^{h-1}}, \delta \in \mathbb{N}^n, \delta_i \in \{0, 1\}, \forall i\}$ forms a $\Bbbk$-basis for $\mathcal{V}_{\Bbbk}(n)_h$. Furthermore, if $p > 0$, then the set $\mathcal{B} := \{A(\delta, \lambda) \mid A \in \Theta^{\pm}(n), \lambda, \delta \in \mathbb{N}^n, \delta_i \in \{0, 1\}, \forall i\}$ forms a $\Bbbk$-basis for $\mathcal{V}_{\Bbbk}(n)$.*

Proof. Note that there is a bijective map from $\{(\delta, \lambda) \mid \delta \in \mathbb{N}^n, \delta_i \in \{0, 1\}, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$ to $(\mathbb{Z}_{l'p^{h-1}})^n$ defined by sending (δ, λ) to $\overline{\lambda + lp^{h-1}\delta}$. Thus by (5.0.5) and 3.3 we conclude that for $A \in \Theta^{\pm}(n)_h$, $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ and $\delta \in \mathbb{N}^n$

$$A(\delta, \lambda)_\varepsilon = \sum_{\substack{\beta \in \mathbb{N}^n, \, \beta_i \in \{0,1\}, \, \forall i \\ \alpha \in \mathbb{N}^n_{lp^{h-1}}}} \varepsilon^{\delta \cdot (\alpha + lp^{h-1}\beta)} \begin{bmatrix} \alpha + lp^{h-1}\beta \\ \lambda \end{bmatrix}_\varepsilon [\![A + \mathrm{diag}(\overline{\alpha + lp^{h-1}\beta})]\!]_h$$

$$= \sum_{\substack{\beta \in \mathbb{N}^n, \, \beta_i \in \{0,1\}, \, \forall i \\ \alpha \in \mathbb{N}^n_{lp^{h-1}}}} \varepsilon^{\delta \cdot \alpha} \varepsilon^{lp^{h-1}(\delta \cdot \beta - \beta \cdot \lambda)} \begin{bmatrix} \alpha \\ \lambda \end{bmatrix}_\varepsilon [\![A + \mathrm{diag}(\overline{\alpha + lp^{h-1}\beta})]\!]_h.$$

Since l' is even and $(l', p) = 1$ we see that p is an odd prime. This, together with the fact that $\varepsilon^l = -1$, implies that $\varepsilon^{lp^{h-1}} = (-1)^{p^{h-1}} = -1$. Thus for $A \in \Theta^{\pm}(n)_h$, $\lambda \in \mathbb{N}^n_{lp^{h-1}}$ and $\delta \in \mathbb{N}^n$ we have

$$A(\delta, \lambda)_\varepsilon = \sum_{\substack{\beta \in \mathbb{N}^n, \, \beta_i \in \{0,1\}, \, \forall i \\ \alpha \in \mathbb{N}^n_{lp^{h-1}}}} \varepsilon^{\delta \cdot \alpha} (-1)^{\beta \cdot (\delta - \lambda)} \begin{bmatrix} \alpha \\ \lambda \end{bmatrix}_\varepsilon [\![A + \mathrm{diag}(\overline{\alpha + lp^{h-1}\beta})]\!]_h$$

(5.3.1)
$$= \sum_{\beta \in \mathbb{N}^n, \, \beta_i \in \{0,1\}, \, \forall i} \varepsilon^{\delta \cdot \lambda} (-1)^{\beta \cdot (\delta - \lambda)} [\![A + \mathrm{diag}(\overline{\lambda + lp^{h-1}\beta})]\!]_h$$

$$+ \sum_{\substack{\beta \in \mathbb{N}^n, \, \beta_i \in \{0,1\}, \, \forall i \\ \alpha \in \mathbb{N}^n_{lp^{h-1}}, \, \lambda < \alpha}} \varepsilon^{\delta \cdot \alpha} (-1)^{\beta \cdot (\delta - \lambda)} \begin{bmatrix} \alpha \\ \lambda \end{bmatrix}_\varepsilon [\![A + \mathrm{diag}(\overline{\alpha + lp^{h-1}\beta})]\!]_h.$$

From 5.2 we see that for $\lambda \in \mathbb{N}^n_{lp^{h-1}}$,

$$\det(\varepsilon^{\delta \cdot \lambda}(-1)^{\beta \cdot (\delta - \lambda)})_{\delta, \beta \in \mathcal{I}_n} = (-\varepsilon)^{\sum_{\delta \in \mathcal{I}_n} \lambda \cdot \delta}(-2)^{2^n-1} = (-\varepsilon)^{\sum_{\delta \in \mathcal{I}_n} \lambda \cdot \delta}(\varepsilon^l - 1)^{2^n-1} \neq 0,$$

where $\mathcal{I}_n = \{\delta \in \mathbb{N}^n \mid \delta_i \in \{0, 1\} \text{ for } 1 \leqslant i \leqslant n\}$. It follows that the martix $(\varepsilon^{\delta \cdot \lambda}(-1)^{\beta \cdot (\delta - \lambda)})_{\delta, \beta \in \mathcal{I}_n}$ is invertible since $\Bbbk$ is a field. Thus by (5.3.1) we conclude that the set $\mathcal{B}_h$ forms a $\Bbbk$-basis for

$\mathcal{W}_{\hat{k}}(n)_h$ and $\mathcal{V}_{\hat{k}}(n)_h = \mathcal{W}_{\hat{k}}(n)_h$. Now we assume $p = \mathrm{char}\,\hat{k} > 0$. Then $\mathcal{B} = \bigcup_{h \geqslant 1} \mathcal{B}_h$. Since the set $\mathcal{B}_h$ is linear independent for all h, we conclude that the set $\mathcal{B}$ is linear independent. Consequently, the set $\mathcal{B}$ forms a $\hat{k}$-basis for $\mathcal{V}_{\hat{k}}(n)$. $\qquad\square$

We are now ready to prove the main result of this paper.

Theorem 5.4. (1) If l' is odd, then $\ker(\xi) = \langle K_i^l - 1 \mid 1 \leqslant i \leqslant n \rangle$ and hence $U_{\hat{k}}(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n \rangle \cong \mathcal{V}_{\hat{k}}(n)$.

(2) If l' is even and $\hat{k}$ is a field with $p = \mathrm{char}\,\hat{k} > 0$, then ξ is injective and hence $U_{\hat{k}}(n) \cong \mathcal{V}_{\hat{k}}(n)$.

Proof. The assertion (1) can be proved in a way similar to the proof of [15, 4.6]. The assertion (2) follows from 2.3, 5.3 and (5.0.4). $\qquad\square$

Theorem 5.5. (1) If l' is odd, then $\mathfrak{u}_{\hat{k}}(n)_h \cong \mathcal{V}_{\hat{k}}(n)_h \cong \mathscr{K}'(n)_h$ for $h \geqslant 1$.

(2) If l' is even and $\hat{k}$ is a field, then $\widetilde{\mathfrak{u}}_{\hat{k}}(n)_h \cong \mathcal{V}_{\hat{k}}(n)_h \cong \mathscr{K}'(n)_h$ for $h \geqslant 1$.

Proof. If either l' is odd or both l' is even and $\hat{k}$ is a field, then by (4.1.2), 5.1 and 5.3, we deduce that $\mathcal{V}_{\hat{k}}(n)_h \cong \mathscr{K}'(n)_h$. If l' is odd, then $\xi(K_i^l - 1) = 0$ and hence the map $\xi : U_{\hat{k}}(n) \to \widehat{\mathcal{K}}_{\hat{k}}(n)$ induces an algebra homomorphism

$$\bar{\xi} : U_{\hat{k}}(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n \rangle \to \widehat{\mathcal{K}}_{\hat{k}}(n).$$

One can prove that the set $\{E^{(A^+)} \prod_{1 \leqslant i \leqslant n} K_i^{-\lambda_i} \big[\begin{smallmatrix} K_i;0 \\ \lambda_i \end{smallmatrix}\big] F^{(A^-)} \mid A \in \Theta^{\pm}(n)_h, \lambda \in \mathbb{N}^n_{lp^{h-1}}\}$ forms a $\hat{k}$-basis of $\mathfrak{u}_{\hat{k}}(n)_h$ in a way similar to the proof of [17, 6.5]. Thus we may regard $\mathfrak{u}_{\hat{k}}(n)_h$ as a $\hat{k}$-subalgebra of $U_{\hat{k}}(n)/\langle K_i^l - 1 \mid 1 \leqslant i \leqslant n \rangle$. From (5.0.4) we see that $\bar{\xi}(\mathfrak{u}_{\hat{k}}(n)_h) = \mathcal{V}_{\hat{k}}(n)_h$. Thus the restriction of $\bar{\xi}$ to $\mathfrak{u}_{\hat{k}}(n)_h$ yields a surjective algebra homomorphism

$$\bar{\xi}' : \mathfrak{u}_{\hat{k}}(n)_h \twoheadrightarrow \mathcal{V}_{\hat{k}}(n)_h.$$

This, together with 5.4(1), implies that $\mathfrak{u}_{\hat{k}}(n)_h \cong \mathcal{V}_{\hat{k}}(n)_h$. Now we assume l' is even and $\hat{k}$ is a field. Since $\xi(\widetilde{\mathfrak{u}}_{\hat{k}}(n)_h) = \mathcal{V}_{\hat{k}}(n)_h$ by (5.0.4), the restriction of ξ to $\widetilde{\mathfrak{u}}_{\hat{k}}(n)_h$ yields a surjective algebra homomorphism

$$\xi' : \widetilde{\mathfrak{u}}_{\hat{k}}(n)_h \twoheadrightarrow \mathcal{V}_{\hat{k}}(n)_h.$$

From 3.7 and 5.3 we see that ξ' is injective. Consequently, $\widetilde{\mathfrak{u}}_{\hat{k}}(n)_h \cong \mathcal{V}_{\hat{k}}(n)_h$. $\qquad\square$

6. The infinitesimal q-Schur algebras and little q-Schur algebras

Let $\mathcal{S}_{\mathcal{Z}}(n, r)$ be the algebra over $\mathcal{Z}$ introduced in [1, 1.2]. It has a $\mathcal{Z}$-basis $\{[A] \mid A \in \Theta(n, r)\}$ defined in [1], where $\Theta(n, r) = \{A \in \Theta(n) \mid \sigma(A) := \sum_{1 \leqslant i,j \leqslant n} a_{i,j} = r\}$. It is proved in [8, A.1] that the algebra $\mathcal{S}_{\mathcal{Z}}(n, r)$ is isomorphic to the q-Schur algebra introduced in [4, 5]. Let $\mathcal{S}_{\hat{k}}(n, r) = \mathcal{S}_{\mathcal{Z}}(n, r) \otimes_{\mathcal{Z}} \hat{k}$. For $A \in \Theta(n, r)$ let

$$[A]_{\varepsilon} = [A] \otimes 1 \in \mathcal{S}_{\hat{k}}(n, r).$$

Let $\Lambda(n,r) = \{\lambda \in \mathbb{N}^n \mid \sum_{1 \leqslant i \leqslant n} \lambda_i = r\}$ and $\overline{\Lambda(n,r)}_{l'p^{h-1}} = \{\bar{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n \mid \lambda \in \Lambda(n,r)\}$. For $A \in \Theta^{\pm}(n)_h$ and $\overline{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n$ we define the element $[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h \in \mathcal{S}_k(n,r)$ as follows:

$$[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h = \begin{cases} \displaystyle\sum_{\substack{\mu \in \Lambda(n,r-\sigma(A)) \\ \overline{\mu} = \overline{\lambda}}} [A + \mathrm{diag}(\mu)]_\varepsilon & \text{if } \sigma(A) \leqslant r \text{ and } \overline{\lambda} \in \overline{\Lambda(n, r-\sigma(A))}_{l'p^{h-1}}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\widetilde{\mathrm{u}}_k(n,r)_h$ be the k-submodule of $\mathcal{S}_k(n,r)$ spanned by the set $\{[\![A + \mathrm{diag}(\overline{\lambda}), r]\!]_h \mid A \in \Theta^{\pm}(n)_h, \overline{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n\}$. According to [12, 4.8], $\widetilde{\mathrm{u}}_k(n,r)_h$ is a k-subalgebra of $\mathcal{S}_k(n,r)$. Note that the algebra $\widetilde{\mathrm{u}}_k(n,r)_1$ is the little q-Schur algebra introduced in [11, 14]. We will prove in 6.1 that the algebra $\widetilde{\mathrm{u}}_k(n,r)_h$ is a homomorphic image of $\widetilde{\mathrm{u}}_k(n)_h$.

Let $\mathcal{S}_\mathcal{Q}(n,r) = \mathcal{S}_\mathcal{Z}(n,r) \otimes_\mathcal{Z} \mathbb{Q}(v)$. For $A \in \Theta^{\pm}(n)$, $\delta \in \mathbb{Z}^n$ let

$$A(\delta,r) = \sum_{\mu \in \Lambda(n,r-\sigma(A))} v^{\mu \cdot \delta}[A + \mathrm{diag}(\mu)] \in \mathcal{S}_\mathcal{Q}(n,r).$$

According to [1], there is an algebra epimorphism

$$\zeta_r : U_\mathcal{Q}(n) \twoheadrightarrow \mathcal{S}_\mathcal{Q}(n,r)$$

satisfying $\zeta_r(E_i) = E_{i,i+1}(\mathbf{0}, r)$, $\zeta_r(K_1^{j_1} K_2^{j_2} \cdots K_n^{j_n}) = 0(\mathbf{j}, r)$ and $\zeta_r(F_i) = E_{i+1,i}(\mathbf{0}, r)$, for $1 \leqslant i \leqslant n-1$ and $\mathbf{j} \in \mathbb{Z}^n$. It is proved in [9] that $\zeta_r(U_\mathcal{Z}(n)) = \mathcal{S}_\mathcal{Z}(n,r)$. By restriction, the map $\zeta_r : U_\mathcal{Q}(n) \to \mathcal{S}_\mathcal{Q}(n,r)$ induces a surjective algebra homomorphism $\zeta_r : U_\mathcal{Z}(n) \to \mathcal{S}_\mathcal{Z}(n,r)$. The map $\zeta_r : U_\mathcal{Z}(n) \to \mathcal{S}_\mathcal{Z}(n,r)$ induces an algebra homomorphism

$$\zeta_{r,k} := \zeta_r \otimes id : U_k(n) \to \mathcal{S}_k(n,r).$$

Proposition 6.1. *If either l' is odd or both l' is even and k is a field then $\zeta_{r,k}(\widetilde{\mathrm{u}}_k(n)_h) = \widetilde{\mathrm{u}}_k(n,r)_h$.*

Proof. According to [10, 6.7], there is a surjective algebra homomorphism

$$\dot{\zeta}_r : \mathcal{K}_\mathcal{Z}(n) \to \mathcal{S}_\mathcal{Z}(n,r)$$

such that

$$\dot{\zeta}_r([A]) = \begin{cases} [A] & \text{if } A \in \Theta(n,r); \\ 0 & \text{otherwise.} \end{cases}$$

The map $\dot{\zeta}_r$ induces a surjective algebra homomorphism

$$\widehat{\dot{\zeta}}_{r,k} : \widehat{\mathcal{K}}_k(n) \to \mathcal{S}_k(n,r)$$

defined by sending $\sum_{A \in \widetilde{\Theta}(n)} \beta_A [A]_\varepsilon$ to $\sum_{A \in \Theta(n,r)} \beta_A [A]_\varepsilon$. It is easy to see that

(6.1.1) $$\zeta_{r,k} = \widehat{\dot{\zeta}}_{r,k} \circ \xi$$

where ξ is given in (5.0.3). This together with (5.0.4) implies that $\zeta_{r,\hat{k}}(\widetilde{\mathfrak{u}}_{\hat{k}}(n)_h) = \widehat{\dot{\zeta}}_{r,\hat{k}}(\mathcal{V}_{\hat{k}}(n)_h)$. Clearly, for $A \in \Theta^{\pm}(n)_h$ and $\bar{\lambda} \in (\mathbb{Z}_{l'p^{h-1}})^n$, we have $\widehat{\dot{\zeta}}_{r,\hat{k}}(\llbracket A + \mathrm{diag}(\bar{\lambda})\rrbracket_h) = \llbracket A + \mathrm{diag}(\bar{\lambda}), r\rrbracket_h$. Combining these facts with 5.1 and 5.3 gives the result. $\square$

Let $\mathbf{s}_{\hat{k}}(n, r)_h$ be the the infinitesimal q-Schur algebra introduced in [2, 3]. The algebra $\mathbf{s}_{\hat{k}}(n, r)_h$ is a certain $\hat{k}$-subalgebra of the q-Schur algebra $\mathcal{S}_{\hat{k}}(n, r)$. According to [2, 5.3.1], we have the following result.

Lemma 6.2. *The set* $\{[A]_\varepsilon \mid A \in \Theta(n, r)_h\}$ *forms a $\hat{k}$-basis of* $\mathbf{s}_{\hat{k}}(n, r)_h$.

For $h \geqslant 1$ let $\mathbf{s}_{\hat{k}}(n)_h$ be the $\hat{k}$-subalgebra of $U_{\hat{k}}(n)$ generated by the elements $E_i^{(m)}$, $F_i^{(m)}$, $K_j^{\pm 1}$, $\left[\begin{smallmatrix} K_j;0 \\ t \end{smallmatrix}\right]$ for $1 \leqslant i \leqslant n-1$, $1 \leqslant j \leqslant n$, $t \in \mathbb{N}$ and $0 \leqslant m < lp^{h-1}$. We will prove in 6.4 that the algebra $\mathbf{s}_{\hat{k}}(n, r)_h$ is a homomorphic image of $\mathbf{s}_{\hat{k}}(n)_h$.

Lemma 6.3. *Each of the following set forms a $\hat{k}$-basis for* $\mathbf{s}_{\hat{k}}(n)_h$:
(1) $\{E^{(A^+)} \prod_{1 \leqslant i \leqslant n} K_i^{\delta_i} \left[\begin{smallmatrix} K_i;0 \\ \lambda_i \end{smallmatrix}\right] F^{(A^-)} \mid A \in \Theta^{\pm}(n)_h,\ \delta \in \mathbb{N}^n,\ \delta_i \in \{0, 1\},\ \forall i,\ \lambda \in \mathbb{N}^n\}$;
(2) $\{A(\delta, \lambda) \mid A \in \Theta^{\pm}(n)_h,\ \delta, \lambda \in \mathbb{N}^n,\ \delta_i \in \{0, 1\},\ \forall i\}$.

Proof. The assertion can be proved in a way similar to the proof of 3.7. $\square$

Proposition 6.4. *We have* $\zeta_{r,\hat{k}}(\mathbf{s}_{\hat{k}}(n)_h) = \mathbf{s}_{\hat{k}}(n, r)_h$.

Proof. From 6.1.1 we see that

$$\zeta_{r,\hat{k}}(A(\delta, \lambda)) = \widehat{\dot{\zeta}}_{r,\hat{k}}(A(\delta, \lambda)_\varepsilon) = A(\delta, \lambda, r)_\varepsilon$$

for all A, δ, λ, where $A(\delta, \lambda, r)_\varepsilon = \sum_{\mu \in \Lambda(n, r - \sigma(A))} \varepsilon^{\mu \cdot \delta} \left[\begin{smallmatrix} \mu \\ \lambda \end{smallmatrix}\right]_\varepsilon [A + \mathrm{diag}(\mu)]_\varepsilon \in \mathcal{S}_{\hat{k}}(n, r)$. Thus by 6.2 and 6.4 we conclude that

$$\zeta_{r,\hat{k}}(\mathbf{s}_{\hat{k}}(n)_h) = \mathrm{span}_{\hat{k}}\{A(\delta, \lambda, r)_\varepsilon \mid A \in \Theta^{\pm}(n)_h,\ \delta, \lambda \in \mathbb{N}^n,\ \delta_i \in \{0, 1\},\ \forall i\} \subseteq \mathbf{s}_{\hat{k}}(n, r)_h.$$

On the other hand, for $A \in \Theta^{\pm}(n)_h$ and $\mu \in \Lambda(n, r - \sigma(A))$ we have $[A + \mathrm{diag}(\mu)] = A(\mathbf{0}, \mu, r) \in \zeta_{r,\hat{k}}(\mathbf{s}_{\hat{k}}(n)_h)$. This implies that $\mathbf{s}_{\hat{k}}(n, r)_h \subseteq \zeta_{r,\hat{k}}(\mathbf{s}_{\hat{k}}(n)_h)$. The assertion follows. $\square$

References

[1] A. A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of GL_n*, Duke Math. J. **61** (1990), 655–677.

[2] A. G. Cox, *On some applications of infinitesimal methods to quantum groups and related algebras, Ph.D thesis*, University of London 1997.

[3] A. G. Cox, *On the blocks of the infinitesimal Schur algebras*, Quart. J. Math **51** (2000), 39–56.

[4] R. Dipper and G. James, *The q-Schur algebra*, Proc. London Math. Soc. **59** (1989), 23–50.

[5] R. Dipper and G. James, *q-Tensor spaces and q-Weyl modules*, Trans. Amer. Math. Soc. **327** (1991), 251–282.

[6] S. R. Doty, D. K. Nakano and K. M. Peters, *On Infinitesimal Schur algebras,* Proc. London Math. Soc. **72** (1996), 588–612.

[7] C. M. Drupieski, *Representations and cohomology for Frobenius–Lusztig kernels*, J. Pure Appl. Algebra **215** (2011), 1473–1491.

[8] J. Du, *Kazhdan–Lusztig bases and isomorphism theorem for q-Schur algebras*, Comtemp. Math. **139**(1992), 121–140.

[9] J. Du, *A note on the quantized Weyl reciprocity at roots of unity,* Alg. Colloq. **2**(1995), 363–372.

[10] J. Du and Q. Fu, *Quantum $\mathfrak{gl}_\infty$, infinite q-Schur algebras and their representations*, J. Algebra **322** (2009) 1516–1547.

[11] J. Du, Q. Fu and J.-P. Wang, *Infinitesimal quantum $\mathfrak{gl}_n$ and little q-Schur algebras,* J. Algebra **287** (2005), 199–233.

[12] J. Du, Q. Fu and J.-P. Wang, *Representations of little q-Schur algebras*, Pacific J. Math., **257**(2012), 343–378.

[13] Q. Fu, *A comparison between little and infinitesimal q-Schur algebras*, Comm. Algebra, **33** (2005), 2663–2682.

[14] Q. Fu, *Little q-Schur algebras at even roots of unity*, J. Algebra, **311**(2007), 202–215.

[15] Q. Fu, *BLM realization for the integral form of quantum $\mathfrak{gl}_n$*, preprint, arXiv:1211.0671.

[16] M. Jimbo, *A q-analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebras, and the Yang-Baxter equation*, Lett. Math. Phy. **11** (1986), 247–252.

[17] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras,* J. Amer. Math. Soc. **3** (1990), 257–296.

[18] G. Lusztig, *Canonical bases arising from quantized enveloping algebras,* J. Amer. Math. Soc. **3** (1990), 447–498.

[19] C. M. Ringel, *Hall algebras and quantum groups*, Invent. Math. **101** (1990), 583–592.

Department of Mathematics, Tongji University, Shanghai, 200092, China.

E-mail address: `q.fu@tongji.edu.cn`